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New dominating sets in social networks

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Abstract Motivated by applications in social networks, a new type of dominating set has been studied in the literature. In this paper, we present results regarding the complexity and approximation in general graphs.

Keywords Dominating set · Greedy approximation · Social network

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1 Introduction

Given a graph $G = (V, E)$, a vertex subset $D \subseteq V$ is called a *dominating set* if every node not in D has a neighbor in D. A dominating set is said to be *connected* if it induces a connected subgraph. The dominating set and the connected dominating set have many applications in networks [7,8,11,12]. Recently, motivated by applications in social networks, Zou et al. [14] proposed to study dominating set D satisfying the following property: (*) For every node v not in D, at least the half of v's neighbors are in D. For simplicity of statement, we will call it (*)-dominating set. There are two problems regarding the (*)-dominating set:

DS-(*): Given a graph $G = (V, E)$, find a (*)-dominating set with minimum cardinality.
CDS-(*): Given a connected graph $G = (V, E)$, find a connected (*)-dominating set with minimum cardinality.

Zou et al. [14] showed that DS-(*) is NP-hard. In this paper, we further show that both DS-(*) and CDS-(*) are APX-hard. We also present greedy approximations for them with performance ratio $O(\ln \delta)$, where δ is the maximum vertex degree of the input graph.

2 Complexities of DS-(*) and CDS-(*)

Theorem 1 *Both DS-(*) and CDS-(*) are APX-hard.*

Proof To show our theorem, we employ an APX-complete problem [3,5] as follows.

VC-3: Given a cubic graph $G = (V, E)$, i.e., a graph in which the degree of every vertex is exactly three, find a vertex-cover with minimum cardinality, where a vertex-cover is a subset C of vertices such that every edge has at least one endpoint in C.

We show that VC-3 is L-reducible to DS-(*) and CDS-(*). To do so, for a cubic graph $G = (V, E)$, we first construct a bipartite graph $H = (V, E, F)$ where $(v, e) \in F$ if and only if v is an endpoint of e in G. Next, we add to H six additional vertices a_i, b_i for $i = 1, 2, 3$ and the following edges:

- (a_i, b_i) for $i = 1, 2, 3$,
- (a_1, e) for all $e \in E$,
- $(a_1, v), (a_2, v), (a_3, v)$ for all $v \in V$.

Denote by G' the obtained graph (Fig. 1). We claim that G has a vertex cover of size at most k if and only if G' has a (*)-dominating set of size at most $k + 3$. To show our claim,

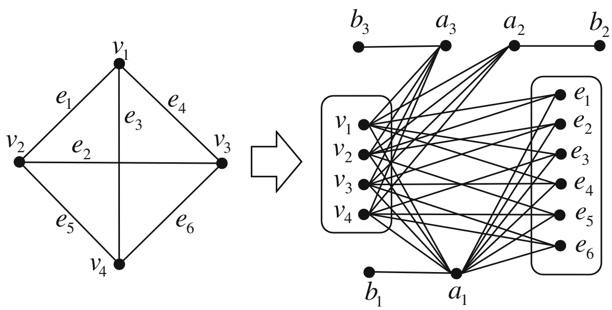

Fig. 1 Construction from G to G'

we first suppose G has a vertex cover C of size k. Let $D = C \cup \{a_i \mid i = 1, 2, 3\}$. In the following we verify that D is a (*)-dominating set for G'.

1. Each b_i has only one neighbor $a_i \in D$.
2. Each $e = (u, v) \in E$ has three neighbors a_1, u and v; among them, $a_1 \in D$ and at least one of u and v is in D because C is a vertex-cover for G.
3. Each $v \in V - C$ has six neighbors in which, $a_1, a_2, a_3 \in D$.

Conversely, suppose D is a (*)-dominating set with size $k+3$ for G'. Note that if $b_i \notin D$, then $a_i \in D$. In the case that $b_i \in D$ and $a_i \notin D$, we may replace b_i by a_i and the resulting set $(D \setminus \{b_i\}) \cup \{a_i\}$ is still a (*)-dominating set of size at most $k + 3$. Therefore, we may assume without loss of generality that $b_i \notin D$ and $a_i \in D$ for $i = 1, 2, 3$. If there is a $e = (u, v) \in E$ belonging to D, then we may replace e by u. The resulting vertex subset is still a (*)-dominating set with size at most $k + 3$. Therefore, we may assume without loss of generality that no $e \in E$ belongs to D. Set $C = D \setminus \{a_1, a_2, a_3\}$. Then, $C \subseteq V$ and $|C| \leq k$. Note that each $e = (u, v) \in E$ has three neighbors a_1, u and v. Since e has to satisfy (*), we must have either $u \in C$ or $v \in C$, that is, C is a vertex-cover of G. This completes the proof of our claim.

Now, suppose G has a minimum vertex-cover of size opt_{VC-3}. Then by the claim, G' has a (*)-dominating set of size $opt_{VC-3} + 3$. Actually this (*)-dominating set is minimum. Otherwise, suppose G' has a (*)-dominating set of size smaller than $opt_{VC-3} + 3$. By the claim, G would have a vertex-cover smaller than opt_{VC-3}, a contradiction. Hence, the minimum size of (*)-dominating set for G' is

$$opt_{DS-(*)} = opt_{VC_3} + 3 \leq 4opt_{VC-3}.$$

Moreover, let D be a (*)-dominating set of size k' for G'. Then from the proof of our claim, we can construct a vertex-cover C of size at most $k' - 3$. Therefore,

$$|C| - opt_{VC_3} \leq k' - (opt_{VC_3} + 3) = |D| - opt_{DS-(*)}.$$

Therefore, VC-3 is L-reducible to DS-(*), and DS-(*) is APX-hard.

Note that in the above proof, every $v \in D \cap V$ is connected to a_1, a_2, a_3 and hence all vertices in D are connected. Therefore, with the same argument, we can show that CDS-(*) is APX-hard. □

3 Approximation algorithms for DS-(*) and CDS-(*)

To present approximation algorithms for DS-(*) and CDS-(*), we first recall some theoretical results about submodular functions and the submodular cover problem.

Consider a finite set X and a real value function f defined on 2^X, all subsets of X. f is called a *submodular* function if for every two subsets A and B,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

f is *monotone increasing* if

$$A \subset B \Rightarrow f(A) \leq f(B).$$

f is called a *polymatroid function* if f is monotone increasing and submodular with $f(\emptyset) = 0$. For an element $x \in X$ and a subset $A \subseteq X$, define $\Delta_x f(A) = f(A \cup \{x\}) - f(A)$. The following can be found in [2,4].

Lemma 1 *f is submodular if and only if for $x \notin B$*

$$A \subset B \Rightarrow \Delta_x f(A) \geq \Delta_x f(B).$$

Consider a polymatroid function f, define

$$\Omega(f) = \{A \mid f(A) = f(X)\}.$$

Let c be a nonnegative cost function defined on X. Denote $c(A) = \sum_{x \in A} c(A)$ for $A \subseteq X$. The following minimization problem is called the *submodular cover* problem (SCP).

SCP: $\min_{A \in \Omega(f)} c(A)$.

Consider the following greedy algorithm for SCP.

Greedy Algorithm: $A \leftarrow \emptyset$;
while= $f(A) < f(X)$ **do**
choose $x \in X$ to maximize $\frac{\delta_x f(A)}{c(x)}$
and set $A \leftarrow A \cup \{x\}$;
output A.

The submodular function and the greedy algorithm have a close relationship [1,10,13]. The following is a very well-known one [13].

Theorem 2 *Let f be a polymatroid integer function on 2^X and c be a nonnegative cost function on X. Then, the* **Greedy Algorithm** *produces an approximate solution for SCP within a factor of $H(\gamma)$ from optimal where $\gamma = \max_{x \in X} f(\{x\})$ and $H(\cdot)$ is the harmonical function, i.e., $H(\gamma) = \sum_{i=1}^{\gamma} \frac{1}{i}$.*

Now, we come back to DS-(*). Consider a graph $G = (V, E)$. For $v \in V$, define $h(v) = \lceil deg(v)/2 \rceil$. For $A \subseteq V$, define $n_A(v) = |\{u \mid (u, v) \in E, u \in A\}|$ and

$$g(A) = \sum_{v \in V} h(v) - \sum_{v \in V - A} p_v(A),$$

where

$$p_v(A) = \max(0, h(v) - n_A(v)).$$

Let us first show some properties of $p_v(A)$.

Lemma 2 *$p_v(A)$ for any fixed $v \in V$ is monotone decreasing and supmodular, i.e., $-p_v(A)$ for any fixed $v \in V$ is monotone increasing and submodular with respect to A.*

Proof It is trivial to see that $p_v(A)$ is monotone decreasing. Next, we consider subsets A and B with $A \subset B$, and a vertex $x \in V - B$. To show the submodularity of $-p_v(\cdot)$, it suffices to prove $-\Delta_x p_v(A) \geq -\Delta_x p_v(B)$. Now, we divide all possibilities into the following two cases:

Case 1 $n_{A\cup\{x\}}(v) \le h(v)$. In this case,

$$\begin{aligned}
-\Delta_x p_v(A) &= \max(0, h(v) - n_A(v)) - \max(0, h(v) - n_{A\cup\{x\}}(v)\\
&= (h(v) - n_A(v)) - (h(v) - n_{A\cup\{x\}}(v))\\
&= n_{A\cup\{x\}}(v) - n_A(v)\\
&= n_x(v)\\
&= n_{B\cup\{x\}}(v) - n_B(v)\\
&\ge \max(0, h(v) - n_B(v)) - \max(0, h(v) - n_{B\cup\{x\}}(v)\\
&= -\Delta_x p_v(B).
\end{aligned}$$

Case 2 $n_{A\cup\{x\}}(v) > h(v)$. In this case, $n_A(v) \ge h(v)$, $n_B(v) \ge h(v)$ and $n_{B\cup\{x\}}(v) \ge h(v)$. Therefore,

$$-\Delta_x p_v(A) = 0 = -\Delta_x p_v(B).$$

□

Now, we show the following.

Lemma 3 *$g(\cdot)$ is a polymatroid function.*

Proof Clearly, $g(\emptyset) = 0$ and g is monotone increasing. To see the submodularity of g, we consider two vertex subsets A and B with $A \subset B$, and a vertex $x \in V - B$. Note that

$$\begin{aligned}
\Delta_x f(A) &= p_x(A) - \sum_{v\in V-A-\{x\}} \Delta_x p_v(A),\\
\Delta_x f(B) &= p_x(B) - \sum_{v\in V-B-\{x\}} \Delta_x p_v(B).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta_x f(A) - \Delta_x f(B) &= (p_x(A) - p_x(B)) - \sum_{v\in B-A} \Delta_x p_v(A)\\
&\quad - \sum_{v\in V-B-\{x\}} (\Delta_x p_v(A) - \Delta_x p_v(B))\\
&\ge 0
\end{aligned}$$

since $p_x(A) \ge p_x(B)$ and $\Delta_x p_v(A) \le 0$ for $p_v(\cdot)$ being monotone decreasing and $\Delta_x p_v(A) \le \Delta_x p_v(B)$ for $p_v(\cdot)$ being supmodular. □

The following lemma indicates that DS-(*) is exactly the SCP with a polymatroid function g and a constant cost function $c(\cdot) = 1$.

Lemma 4 *A vertex subset A is a (*)-dominating set if and only if $g(A) = g(V)$. Moreover, if $g(A) < g(V)$, then there exists a vertex x such that $\Delta_x g(A) > 0$.*

Proof $g(A) = g(V)$ if and only if $p_v(A) = 0$ for all $v \in V - A$ if and only if $n_A(v) \ge h(v)$ for all $v \in V - A$ if and only if A is a (*)-dominating set. Moreover, if $g(A) < g(V)$, then there exists $v \in V - A$ such that $n_A(v) < h(v)$. Choose $x = v$. We have $\Delta_x g(A) > 0$. □

Now, we apply the Greedy Algorithm to compute approximation solutions for DS-(*). By Theorem 2, we have the following result.

Theorem 3 *There exists a greedy approximation for DS-(*) with performance ratio* $H(\lceil(3/2)\delta\rceil)$ *where* δ *is the maximum vertex degree of input graph.*

Proof $\gamma = \max_{x \in V} g(\{x\}) = \delta + \lceil\delta/2\rceil = \lceil(3/2)\delta\rceil$. □

Next, we study CDS-(*) and consider a connected graph $G = (V, E)$. For any vertex subset $A \subseteq V$, denote by $q(A)$ the number of connected components of the subgraph $G[A]$ induced by A. Define

$$f(A) = g(A) - q(A).$$

Then, f has the following properties.

Lemma 5 *f is monotone increasing.*

Proof Let $x \in L - A$. Since g is monotone increasing, we have $g(A) \leq g(A \cup \{x\})$. Next, compare $q(A \cup \{x\})$ with $q(A)$. Clearly, if x has a neighbor in A, then $q(A \cup \{x\}) \leq q(A)$. Therefore, $f(A) \leq f(A \cup \{x\})$. If x has no neighbor in A, then $q(A \cup \{x\}) = q(A) + 1$. However, in this case, $g(A) < g(A \cup \{x\})$ because

- $n_A(x) = 0$ and $g(A)$ has a term $-p_x(A) = -h(x) < 0$,
- $g(A \cup \{x\})$ has a term $-p_x(A \cup \{x\})$, and
- for $v \in L - A - \{x\}$, $-p_v(A) \leq -p_v(A \cup \{x\})$.

Therefore, $f(A) \leq f(A \cup \{x\})$. □

Lemma 6 *A is a connected (*)-dominating set if and only if $f(A) = f(V)$.*

Proof If A is a connected (*)-dominating set, then $g(A) = g(V)$ and $q(A) = q(V)$. Hence $f(A) = f(V)$. Conversely, suppose $f(A) = f(V)$. If $q(A) > q(V)$, then $g(A) > g(V)$ which is impossible. Therefore, $q(A) = q(V)$ and $g(A) = g(V)$, which implies that A is a (*)-dominating set. □

By Lemma 5, $\Omega(f) = \{A \subseteq V \mid f(A) = f(V)\}$ is a well-defined family of vertex subsets which reaches the maximum value of f. By Lemma 6, CDS-(*) is in the form of SCP. Can we apply the GREEDY ALGORITHM with potential function f to compute an approximation solution for CDS-(*)? The answer is NO. The reason is that when $f(A) < f(V)$, we may not be able to find a vertex $x \in V - A$ such that $\Delta_x f(A) > 0$. A counterexample is as follows: Let G be a path $(v_1, v_2, \ldots, v_6)$ and $A = \{v_2, v_5)$. Then $f(A) = 4 < 5 = f(V)$. However, no vertex x makes $\Delta_x f(A) > 0$. To overcome this trouble, we introduce another function $s(A)$ for $A \in 2^V$. $s(A)$ is the number of connected components of the subgraph $G\{A\}$ with vertex set V and edge set $\{e \in E \mid e \cap A \neq \emptyset\}$. The following can be found in [2,9].

Lemma 7 *$|V| - s(\cdot)$ is a polymatroid function on 2^V.*

Define $t(A) = |V| - s(A) + g(A) + q(A)$ for $A \in 2^V$. Now, we have

Lemma 8 *$t(\emptyset) = 0$ and $t(\cdot)$ is monotone increasing.*

Proof It follows immediately from Lemmas 5 and 7. □

Lemma 9 *A is a connected (*)-dominating set if and only if $t(A) = t(V)$. Moreover, if $t(A) < t(V)$, then there exists a vertex x such that $\Delta_x t(A) > 0$.*

Proof Since $f(A)$ and $|V| - s(A)$ are monotone increasing, $t(A) = t(V)$ if and only if $f(A) = f(V)$ and $s(A) = s(V)$ if and only if A is a connected (*)-dominating set. Moreover, if $t(A) < t(V)$, then we have $g(A) < g(V)$ or $q(A) > q(V)$ or $s(A) > s(V)$. If $s(A) > s(V) = 1$, then G_A is not connected. Since G is connected and $G\{A\}$ has vertex set V, there must exist two adjacent connected components of $G\{A\}$, i.e., there is an edge (u, v) connecting two connected components, but, $u, v \in V - A$. Choose $x = u$ or v. We would have $s(A) > s(A \cup \{x\})$. Moreover, by Lemma 5, $f(A) \leq f(A \cup \{x\})$. Hence $t(A) < t(A \cup \{x\})$.

Now, we may assume $s(A) = s(V) = 1$ and $g(A) < g(V)$ or $q(A) > q(V) = 1$. If $q(A) > q(V)$, i.e., the subgraph $G[A]$ induced by A is not connected. Let u and v belong to two different connected components of $G[A]$ and reach the minimum distance among such pairs of vertices. Then, the distance between u and v is two because, otherwise, $s(A) > 1$ contradicting our assumption. This means that u and v must have a neighbor x in common. This x connects two connected components of $G[A]$ into one and hence $q(A) > q(A \cup \{x\})$. Moreover, $g(A) \leq g(A \cup \{x\}$ by Lemma 3. Thus, $t(A) < t(A \cup \{x\})$. Finally, we may assume $s(A) = s(V)$, $q(A) = q(V)$ and $g(A) < g(V)$. Then, by Lemma 4, there exists $x \in V - A$ such that $g(A) < g(A \cup \{x\})$ and hence $t(A) < t(A \cup \{x\})$. □

By Lemmas 7 and 8, we can use the Greedy Algorithm to compute an approximation solution for CDS-(*). However, we cannot apply Theorem 2 to establish its performance ratio because $t(A)$ is not submodular. But, we can employ a technique of Du et al. [2] to give an analysis.

Theorem 4 Greedy Algorithm *with potential function* $t(\cdot)$ *produces an approximation for CDS-(*) within a factor of* $2 + \ln((5/2)\delta)$ *from optimal where* δ *is the maximum vertex degree of input graph.*

Proof Let $x_1, x_2, \ldots, x_g$ be chosen in turn by the Greedy Algorithm. Denote $C_i = \{x_1, x_2, \ldots, x_i\}$. Let $y_1, y_2, \ldots, y_{opt}$ be an optimal solution for CDS-(*), which are ordered such that for any $j = 1, 2, \ldots, opt$, $C_i^* = \{y_1, y_2, \ldots, y_j\}$ induces a connected subgraph. This ordering gives us the following inequality.

$$-\Delta_x q(A \cup C_j^*) \leq -\Delta_x q(A) + 1$$

for any $x \in V$ and $A \in 2^V$. In fact, suppose k is the number of connected components of $G[A]$, adjacent to x. Then, $-\Delta_x q(A) = k - 1$. Since $G[C_j^*]$ is connected, the number of connected components of $G[A \cup C_j^*]$, adjacent to x, is at most $k + 1$. Therefore, $-\Delta_x q(A \cup C_j^*)$ is at most $k = -\Delta_x q(A) + 1$.

Moreover, since $g(A)$ and $|V| - s(A)$ are polymatroid functions, we have

$$\Delta_x t(A) \geq \Delta_x t(A \cup C_j^*) - 1 \tag{1}$$

for any $x \in V$ and $A \in 2^V$.

By the greedy rule of the Greedy Algorithm,

$$\Delta_{x_i} t(C_{i-1}) \geq \Delta_{y_j} t(C_{i-1})$$

for $j = 1, 2, \ldots, opt$, where $C_0 = \emptyset$. Therefore,

$$\begin{aligned}\Delta_{x_i} t(C_{i-1}) &\geq \frac{\sum_{j=1}^{opt} \Delta_{y_j} t(C_{i-1})}{opt} \\ &\geq \frac{-opt + 1 + \sum_{j=1}^{opt} \Delta_x t(C_{i-1} \cup C_j^*)}{opt} \\ &= \frac{-opt + 1 + t(C_{i-1} \cup C_{opt}^*) - t(C_{i-1})}{opt} \\ &= \frac{-opt + 1 + t(V) - t(C_{i-1})}{opt}\end{aligned}$$

Set $a_i = -opt + 1 + t(V) - t(C_i)$. Then

$$a_{i-1} - a_i \geq \frac{a_{i-1}}{opt}.$$

Hence,

$$a_i \leq a_{i-1}\left(1 - \frac{1}{opt}\right) \leq \cdots \leq a_0 \left(1 - \frac{1}{opt}\right)^i.$$

Note that $a_0 = -opt + 1 + t(V) \geq -opt - 1 + 2|V| - 2 \geq opt$ and $a_g = -opt + 1 < opt$. We can choose i such that $a_i \geq opt > a_{i+1}$. Hence,

$$a_g = -opt + 1 \leq opt - (g - i), \tag{2}$$

and

$$opt \leq a_0 \left(1 - \frac{1}{opt}\right)^i. \tag{3}$$

By Eq. 3 and $1 + x \leq e^x$, we have

$$opt \leq a_0 e^{-i/opt},$$

which is equivalent to

$$i \leq opt \cdot \ln \frac{a_0}{opt}.$$

Note that $a_0 = -opt + \sum_{v \in V} h(v) + |V| - 1$. Since $\Delta_x g(A) \leq h(x) + \delta < (3/2)\delta + 1$ for any $x \in V$ and $A \in 2^V$, we have

$$\sum_{v \in V} h(v) \leq opt \cdot ((3/2)\delta + 1).$$

Moreover, since y_1 can dominate at most $\delta + 1$ vertices and y_2 can dominate at most $\delta - 2$ vertices not dominated by y_1, etc, we have

$$|V| \leq opt \cdot (\delta - 1) + 2.$$

Combining above inequalities, we obtain

$$\begin{aligned}a_0 &\leq -opt - 1 + opt \cdot ((3/2)\delta + 1) + opt \cdot (\delta - 1) + 2 \\ &\leq opt \cdot (5/2)\delta.\end{aligned}$$

Therefore,

$$i \le opt \cdot \ln((5/2)\delta).$$

By Eq. 2, we obtain

$$g \le 2opt - 1 + i \le opt \cdot (2 + \ln((5/2)\delta)).$$

□

4 Discussion

The reduction from VC-3 to DS-(*) and CDS-(*) can also be modified to a reduction from the SET- COVER problem to DS-(*) and CDS-(*). This gives a light to show $O(\ln n)$ lower bound for performance ratio of polynomial-time approximation. Indeed, Feige [6] has shown that the SET- COVER problem cannot have polynomial-time $\rho \ln n$-approximation for $\rho < 1$ unless $NP \subseteq DTIME(n^{O(\log \log n)})$. But, in the result of Feige [6], there is no upper bound on the size of subsets and no upper bound on the number of subsets in which an element appears. Therefore, the graph obtained from such a reduction does not have a bounded vertex degree, and the reduction cannot transform the lower bound from SET- COVER to DS-(*) and CDS-(*). Can Faige's lower bound be applied to a special case of SET- COVER with an upper bound for sizes of subsets and an upper bound for the number of subsets in which an element appears? This is an interesting open problem.

It may be possible to improve the performance of the greedy approximation for CDS-(*) by finding a new potential function. For example, if we can find a better way to combine $g(A)$ and $s(A)$, the performance ratio is possibly reduced to $2 + \ln(3\delta/2)$. How do we do it? This is another interesting open problem.

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